Note Title 2/20/2008

WKB approximation

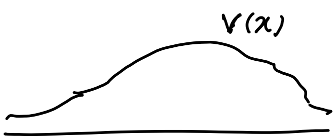
(Wentzel - Kramers - Brillouin 1926)

This a semiclassical approximation to calculate the transmission probability. Recall that TMM was a quantum mechanical approximation, hence, it is more accurate than WKB.

For free particle:

$$\psi(x) = A e^{ikx} \qquad \left(\text{note: } \psi(x,t) = \psi(x) e^{-i\omega t}\right)$$

If there is a potential:



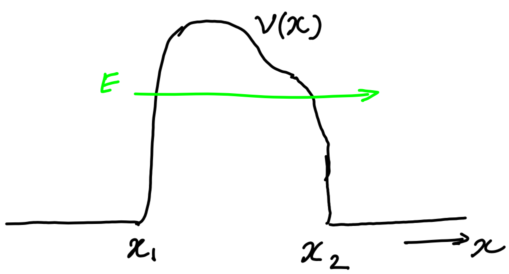
$$E - V = \frac{\hbar^2 k^2}{2m} \qquad E > V \rightarrow k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar}$$

$$V - E = \frac{\hbar^2 k^2}{2m} \qquad E < V \rightarrow k(x) = \frac{\sqrt{2m(V(x) - E)}}{\hbar}$$

$$\psi(x,t) = A e^{ikx} + B e^{-ikx} \rightarrow A e^{i\int k(x)dx} + B e^{-i\int k(x)dx}$$

$$\rightarrow \boxed{\psi_{WKB}(x) = \frac{1}{\sqrt{k(x)}} \left(A e^{i\int k(x)dx} + B e^{-i\int k(x)dx} \right)}$$

Tunneling probability:



$$\boxed{T(E) = e^{-2\int_{x_1}^{x_2} k(x)dx}}$$

where:
$$E - V(x) = \frac{\hbar^2 k^2}{2m} \qquad E > V(x)$$

$$V(x) - E = \frac{\hbar^2 k^2}{2m} \qquad E < V(x)$$

Dirac notation

$$\int \varphi^*(r,t)\,\psi(r,t)\,d^3r = \underset{\text{bra}}{\langle \varphi(r,t)|}\,\underset{\text{ket}}{|\psi(r,t)\rangle}$$

$|\psi(r,t)\rangle$: state vector in Hilbert space

Properties

Conjugate: $\langle \varphi | \psi \rangle^* = \langle \psi | \varphi \rangle$

Operator: $\hat{A}|\psi\rangle$

Schrödinger Equ: $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

Time dependence: $|\psi_n, t\rangle = |\psi_n\rangle e^{-i\omega_n t}$, $E_n = \hbar\omega_n$

bra & ket: $|b\rangle = \hat{A}|a\rangle \Rightarrow \langle b| = \langle a|\hat{A}^\dagger$

$A^\dagger$ ← dagger : Hermitian Conjugate (complex conjugate transpose)

$$\langle a|\hat{A}|b\rangle^* = \langle b|\hat{A}^\dagger|a\rangle$$

Hermitian operator: $A^\dagger = A$ Has real eigenvalues

So A is an operator for a physical observable.

Normalization $\int \varphi_n^* \varphi_m \, d^3r = \delta_{nm}$

$$\langle \varphi_n | \varphi_m \rangle = \delta_{nm}$$

Expansion

If $|n\rangle$'s are the eigenfunctions, an arbitrary state-function can be expanded by $|n\rangle$'s:

$$|\psi\rangle = \sum_n b_n |n\rangle$$

$$\Rightarrow \quad \langle m|\psi\rangle = \sum_n \langle m| b_n |n\rangle = \sum_n b_n \langle m|n\rangle$$

$$= \sum_n b_n \delta_{nm} = b_m$$

$$\Rightarrow \quad b_n = \langle n|\psi\rangle$$